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Reduced order control based on approximate inertial manifolds

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Abstract

A reduced-order method based on approximate inertial manifolds is applied to optimal control problems in infinite-dimensional state spaces. A detailed analysis of the method is given for the linear quadratic regulator problem. The method can also be applied to higher-order control systems with an appropriate decomposition of the state space in terms of slow and fast exponential decay.

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1. Introduction

We consider optimal control problems for control systems governed by partial differential equations. There has been increased research interest in developing

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reduced-order control methods based on the proper orthogonal decomposition approach [1,12,13] and on the reduced-basis method [8–10] for distributed parameter control systems. The key issue for the reduction consists in selecting basis elements which are rich in information in the sense that they capture well the essential dynamical properties of the original control system. After the basis elements are selected the standard Galerkin approach is applied to obtain the reduced order control system. For linear control systems many alternative reduction methods have been proposed and analyzed including the Hankel-norm approximation [5] and the balanced truncation realization [2]. The LQG-balanced truncation method was introduced in the finite-dimensional literature by [11] and other interpretations followed in [14,16]. The infinite-dimensional theory was developed in [2]. It basically involves solving two Riccati equations and then performing the Hankel singular-value decomposition. It is related to the method proposed here in such a manner that it provides an effective way to obtain the decomposition of systems in the context of the inertial manifold approach. Furthermore our reduction method can be used to perform the LQG-balanced truncation for the infinite-dimensional system following the approximation procedure discussed in [2]. The selection of references here is by no means complete and is based on its relevance to our presentation. For further references we refer to the citations in the quoted references.

In this paper we discuss an order-reduction method based on approximate inertial manifolds. Specifically we use the linear quadratic regulator problem as an important benchmark problem and provide a detailed analysis on the closed-loop behavior of the resulting reduced-order control synthesis. An important feature of inertial manifolds consists in the description of the small-scale dynamics as a graph of the large-scale dynamics. The inertial manifold is a finite-dimensional invariant manifold that attracts all orbits exponentially. Thus it is natural to expect that controlling the inertial manifold dynamics results in the ability to control the full underlying control system. In the case of linear systems this approach results in a modal control method based on an open-loop invariant subspace. There are some technical difficulties associated with the approach: (1) the existence of the inertial manifold can only be proved for a limited class of systems, (2) the construction of the inertial manifold is highly involved and technical, (3) the analysis on the behavior of the closed-loop system has not been fully addressed.

In this paper we therefore consider an approximate manifold that is not necessary invariant under the dynamics but approximates with any desired accuracy all orbits starting from a bounded set. We use the nonlinear Galerkin approximation as proposed e.g. in [4,15], and the manifold is represented by a stationary graph determined by the residual dynamics. If one would simply truncate the residual dynamics (the flat manifold), then this approach coincides with the standard Galerkin approximation. Moreover we may also construct a higher order manifold based on Picard-type iterates. This paper especially focuses on the linear quadratic regulator problem for which we demonstrate the effectiveness of the approximate inertial manifold approach. The detailed analysis for the nonlinear case will be reported on elsewhere. For the linear quadratic regulator problem the method can be formulated as finding invariant subspaces of the corresponding Hamiltonian operator. As will be discussed in Section 3 this leads to a very efficient algorithm for constructing a stabilizing nearly optimal linear feedback synthesis. In summary the approximate inertial manifold approach can improve the existing reduction methods in the following manner. First, it constructs a manifold that captures the property of the original dynamics beyond the one based on reduced order method/Galerkin approximations. Second, it is possible to further reduce the order of the existing reduced order systems themselves, by applying the inertial manifold technique to the reduced order system.

The outline of the paper is as follows. In Section 2 we describe the general inertial manifold approach and establish an upper bound estimate for the reduced-order control in terms of the specified performance index. Section 3 is devoted to the linear quadratic regulator (LQR-) problem and the reduced-order algorithm for the construction of the optimal feedback gain operator. In Section 4 we present a numerical example and demonstrate the applicability of our approach.

2. Approximate inertial manifold and reduced-order control system

Let X and U be separable Hilbert spaces. We consider the optimal control problem

$$\min J(x,u) = \int_0^\infty (\ell(x(t)) + h(u(t))\mathrm{d}t \tag{2.1}$$

over admissible U-valued measurable controls u on $(0,\infty)$, subject to

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A_0 x(t) + F(x(t)) + Bu(t), \quad x(0) = x_0 \in X, \tag{2.2}$$

where A_0 is a negative self-adjoint operator in X. We assume that (2.2) is a well-posed control system, i.e, given any $x_0 \in X, T > 0$ and $u \in L^2(0,T;U)$, there exists a unique weak solution $x = x(t; x_0, u) \in C(0,T;X)$ to (2.2) which depends continuously on $(x_0, u) \in X \times L^2(0,T;U)$. For example we can formulate the control problem (2.1), (2.2) in a Gelfand triple formulation or as semi-linear control systems, see e.g. [7].

We assume that $-A_0$ has eigen-pairs (λ_i, ϕ_i) in ascending order and that $\{\phi_i\}_{i=1}^\infty$ forms an orthonormal basis of X. Let P_1 be the orthogonal projection of X onto $X_1 = \mathrm{span}\{\phi_i : 1 \leqslant i \leqslant N\}$. Expressing x as $x = x_1 + x_2$, with $x_1(t) = P_1 x(t)$ and $x_2(t) = P_2 x(t)$, where $P_2 = I - P_1$, we have from (2.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = A_0 x_1(t) + P_1 F(x_1(t) + x_2(t)) + P_1 Bu(t),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = A_0 x_2(t) + P_2 F(x_1(t) + x_2(t)) + P_2 Bu(t).$$

In the linear Galerkin approach the higher order modes x_2 are neglected resulting in the control system

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_1(t) = A_0\hat{x}_1(t) + P_1F(\hat{x}_1(t)) + P_1Bu(t).$$

In the nonlinear Galerkin approach $\frac{\mathrm{d}}{\mathrm{d}t}x_2(t)$ is assumed to be negligible compared to $A_0x_2(t)$. This suggests to consider the control system

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_1(t) = A_0\hat{x}_1(t) + P_1F(\hat{x}_1(t)+\hat{x}_2(t)) + P_1Bu(t),\\ &A_0\hat{x}_2(t) + P_2F(\hat{x}_1(t)+\hat{x}_2(t)) + P_2Bu(t) = 0,\end{aligned} \tag{2.3}$$

or equivalently, $\hat{x} = \hat{x}_1 + \hat{x}_2$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}P_1\hat{x}(t) = A_0\hat{x}(t) + F(\hat{x}(t)) + Bu(t). \tag{2.4}$$

We assume that given $\hat{x}_1 \in X_1 = P_1X$ and $u \in U$ the second equation in (2.3) has a unique solution $\hat{x}_2 \in X_2 = P_2X$, and that this defines a Lipschitz continuous mapping $\hat{x}_2 = \Phi(\hat{x}_1, u)$ from $X_1 \times U$ into X_2. In this way we obtain the finite-dimensional control system in X_1

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = A_0z(t) + P_1F(z(t) + \Phi(z(t), u(t))) + P_1Bu(t), \quad z(0) = P_1x_0, \tag{2.5}$$

and the corresponding reduced-order optimal control problem

$$\min \int_0^\infty \ell(z(t) + \Phi(z(t), u(t))) + h(u(t))\mathrm{d}t \tag{2.6}$$

subject to (2.5).

Next we discuss an error estimate for the solutions of the reduced-order equation. Let $\langle\cdot,\cdot\rangle$ be the duality product on $\mathrm{dom}((-A_0)^{-1/2}) \times \mathrm{dom}((-A_0)^{1/2})$. We assume that $B \in \mathscr{L}(U, \mathrm{dom}((-A_0)^{-1/2})$ and

$$\langle F(x) - F(y), x - y\rangle \leqslant c|x-y|_X^2 \quad \text{for } x, y \in \mathrm{dom}((-A_0)^{1/2}). \tag{2.6}$$

In particular, the assumption on B is satisfied if $B \in \mathscr{L}(U, X)$, and (2.6) holds with $c = 0$, if F is conservative.

Also, we assume that $P_2B = 0$ and thus $\hat{x}_2 = \Phi(\hat{x}_1)$. For $u \in L^2(0, T; U)$ the solution z to (2.5) satisfies $z \in H^1(0, T; X_1)$ and for the solution to (2.1) we have $x \in L^2(0, T; \mathrm{dom}((-A_0)^{1/2}) \cap H^1(0, T; \mathrm{dom}((-A_0)^{-1/2})$. Note that from (2.4), $\hat{x}(t) = \hat{x}_1(t) + \hat{x}_2(t)$ (with $z(t) = \hat{x}_1(t)$) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = A_0\hat{x}(t) + F(\hat{x}(t)) + Bu(t) + P_2\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\hat{x}_1(t)).$$

Thus, $e(t) = x(t) - \hat{x}(t) \in X$ satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|e(t)|_X^2 = \langle F(x(t)) - F(\hat{x}(t)), x(t) - \hat{x}(t)\rangle$$

$$+ \langle A_0 e(t), e(t)\rangle + \left\langle P_2 \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\hat{x}_1(t)), e(t)\right\rangle$$
$$\leqslant c|e(t)|_X^2 + \frac{1}{2}\langle A_0 e(t), e(t)\rangle + \frac{1}{2}\left|(-A_0)^{-1/2} P_2 \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\hat{x}_1(t))\right|^2. \tag{2.7}$$

Note that for all $\phi \in X$

$$|(-A_0)^{-1/2} P_2 \phi|_X \leqslant \frac{1}{\sqrt{\lambda_{N+1}}}|P_2\phi|_X.$$

By Gronwall's inequality we obtain the following estimate for the nonlinear Galerkin approximation

$$|e(t)|_X^2 + \int_0^t |\exp(2c(t-s))(-A_0)^{1/2} e(s)|_X^2 \,\mathrm{d}s$$
$$\leqslant \exp(2ct)|e(0)|_X^2 + \int_0^t \exp(2c(t-s))\frac{1}{\lambda_{N+1}}\left|P_2 \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\hat{x}_1(s))\right|_X^2 \mathrm{d}s. \tag{2.8}$$

In order to carry out the error estimate beyond (2.8) one needs to determine and exploit the regularity of the map Φ. We refer to [4,15] for such results.

To demonstrate the use of the a-priori estimate (2.8) in the context of the optimal control problem (2.1), (2.2), let u and u^* be optimal controls to (2.5), (2.6) and (2.1), (2.2), respectively. Moreover, let x and x^* in X be the solution to (2.1) with u and u^*, respectively and z^* and z the solution to (2.6), respectively. Then for the finite horizon control on $[0, T]$ we have

$$\begin{aligned} 0 \leqslant J(x,u) - J(x^*,u^*) &= J(x,u) - J(z,u) + J(z,u) \\ &\quad - J(z^*,u^*) + J(z^*,u^*) - J(x^*,u^*) \\ &\leqslant J(x,u) - J(z,u) + J(z^*,u^*) - J(x^*,u^*) \end{aligned}$$

where we used $J(z,u) - J(z^*,u^*) \leqslant 0$. Thus

$$0 \leqslant J(x,u) - J(x^*,u^*) \leqslant C(|x^* - z^*|_{L^2(0,T;X)} + |x - z|_{L^2(0,T;X)}).$$

That is, the error estimate of e provides an upper bound estimate of the performance of the suboptimal control u based on the reduced-order control problem (2.5), (2.6).

The closed-loop behavior of the resulting reduced-order control synthesis is described for the linear quadratic regulator problem in the next section.

3. LQR problem and reduced order algorithm

Consider the linear control system

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + Bu(t), \tag{3.1}$$

where A is the infinitesimal generator of a C_0 semigroup on X and $B \in \mathscr{L}(U, X)$. The LQR problem is the optimal control problem of minimizing the quadratic cost

$$\int_0^\infty (Qx(t), x(t))_X + |u(t)|^2 \, \mathrm{d}t \tag{3.2}$$

over $u \in L^2(0, \infty; U)$ subject to (3.1), where Q is a bounded, self-adjoint, nonnegative operator on X. Let $X_1 = \operatorname{span}\{\phi_i : 1 \leqslant i \leqslant N\}$ with $\phi_i \in \operatorname{dom}(A)$ orthonormal and let P_1 be the orthogonal projection of X onto X_1. Set $P_2 = I - P_1$. The orthonormal basis can be generated by eigen-functions of A_0 with $A = A_0 + A_1$, by proper orthogonal decomposition, or by reduced-basis methods, for example. For the reduced-basis method the orthonormal basis can be constructed from singular value decomposition of the mass matrix of the reduced-basis. With X_1 fixed we arrive at the following partitions

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22}, \end{bmatrix}$$

on $X = X_1 \times X_2$ where $X_1 = P_1 X$ and $X_2 = P_2 X$.

Let $\{S_2(t)\}$ be the semigroup generated by A_{22} on X_2 and we assume that it is exponentially stable. Then from (3.1) we have

$$\begin{aligned} x_2(t) &= S_2(t)x_2(0) + \int_0^t S_2(t-s)(A_{21}x_1(s) + B_2u(s))\mathrm{d}s \\ &= S_2(t)(x_2(0) + A_{22}^{-1}(A_{21}x_1(0) + B_2u(0)) - A_{22}^{-1}(A_{21}x_1(t) + B_2u(t))) \\ &\quad + \int_0^t S_2(t-s)A_{22}^{-1}\left(A_{21}\frac{\mathrm{d}}{\mathrm{d}t}x_1(s) + B_2\frac{\mathrm{d}}{\mathrm{d}t}u(s)\right)\mathrm{d}s. \end{aligned} \tag{3.3}$$

The approximate manifold is given by

$$x_2 = -A_{22}^{-1}(A_{21}x_1 + B_2u).$$

It results from truncating the transient time and higher order terms in (3.3). Then we obtain, assuming that $B_2 = 0$, the reduced order LQR problem

$$\min \int_0^\infty (\tilde{Q}_{11}x_1, x_1)_{X_1} + |u(t)|^2 \, \mathrm{d}t \tag{3.4}$$

subject to

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = \tilde{A}_{11}x_1(t) + \tilde{B}_1u(t) \tag{3.5}$$

and

$$\begin{aligned} &\tilde{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \\ &\tilde{Q}_{11} = [I_1 - (A_{22}^{-1}A_{21})^*]Q\begin{bmatrix} I_1 \\ -A_{22}^{-1}A_{21} \end{bmatrix}, \quad \tilde{B}_1 = B_1. \end{aligned}$$

where I_1 is the $N \times N$ identity matrix.

Assume now that (A, B) is stabilizable and that (A, Q) is detectable. Then there exist operators $K \in \mathscr{L}(X, U)$ and $G \in \mathscr{L}(X)$ such that $A - BK$ and $A - QG$ generate exponentially stable C_0 semigroups on X. The optimal control of (3.1), (3.2) is given in the feedback form $u(t) = -B^*\Pi x(t)$ where the bounded, self-adjoint operator Π is the unique nonnegative solution to the algebraic Riccati equation

$$A^*\Pi x + \Pi Ax - \Pi BB^*\Pi x + Qx = 0$$

for every $x \in \mathrm{dom}(A)$. Here A^*, B^* are the Hilbert space adjoints of A and B, respectively and $\Pi x \in \mathrm{dom}(A^*)$ for $x \in \mathrm{dom}(A)$. This result on LQR problem is standard, we refer e.g. to, [3,6]. Equivalently if we let $p(t) = \Pi x(t)$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t), p(t))^t = H(x(t), p(t))^t,$$

where the Hamiltonian operator H in $X \times X$ is given by

$$H = \begin{bmatrix} A & -BB^* \\ -Q & -A^* \end{bmatrix}.$$

If λ is a closed loop eigenvalue of $A - BB^*\Pi$, then λ is an eigenvalue of H. In fact, if $x \in \mathrm{dom}(A)$ satisfies $(A - BB^*\Pi)x = \lambda x$, then $\Pi x \in \mathrm{dom}(A^*)$ and

$$-Qx - A^*\Pi x = \Pi(A - BB^*\Pi)x = \lambda\Pi x.$$

Thus, λ is an eigenvalue of H and the corresponding eigenfunction is given by $(x, \Pi x)$.

Let the eigenfunction $(x, p) = (x, \Pi x) \in X \times X$ be decomposed as $x = (x_1, x_2)$, $p = (p_1, p_2) \in X_1 \times X_2$. Then

$$\begin{aligned} x_2 &= (I_2 - \lambda A_{22}^{-1})^{-1} A_{22}^{-1}(-A_{21}x_1 + B_2B_1^* p_1 + B_2B_2^* p_2), \\ p_2 &= (I_2 + \lambda A_{22}^{-*})^{-1} A_{22}^{-*}(-A_{12}^* p_1 - Q_{21}x_1 - Q_{22}x_2). \end{aligned}$$

We assume now that $\|A_{22}^{-1}B_2\|$ and $\|A_{22}^{-1}Q_{22}\|$ are significantly smaller than $\|A_{22}^{-1}A_{21}\|$ and $\|A_{12}A_{22}^{-1}\|$ and consequently choose $B_2 = 0$ and $Q_{22} = 0$ in the following calculations. If we approximate $(I_2 - \lambda A_{22}^{-1})^{-1}$ and $(I_2 + \lambda A_{22}^{-*})^{-1}$ by I_2, then we obtain the reduced order problem

$$H_{11}(x_1, p_1) = \lambda(x_1, p_1),$$

where H_{11} is the reduced-order Hamiltonian for (3.4), (3.5) in $X_1 \times X_1$

$$H_{11} = \begin{bmatrix} \tilde{A}_{11} & -\tilde{B}_1\tilde{B}_1^* \\ -\tilde{Q}_{11} & -\tilde{A}_{11}^* \end{bmatrix}. \tag{3.6}$$

If we approximate $(I_2 - \lambda A_{22}^{-1})^{-1}$ by $I_2 + \lambda A_{22}^{-1}$ and $(I_2 + \lambda A_{22}^{-*})^{-1}$ by $I_2 - \lambda\, A_{22}^{-*}$, we obtain the generalized eigenvalue problem

$$H_{11}\begin{pmatrix} x_1 \\ p_1 \end{pmatrix} = \lambda \begin{bmatrix} I + A_{12}A_{22}^{-2}A_{21} & 0 \\ 0 & I + (A_{12}A_{22}^{-2}A_{21})^* \end{bmatrix} \begin{pmatrix} x_1 \\ p_1 \end{pmatrix}. \tag{3.7}$$

This is equivalent to the following: From (3.3)

$$\begin{aligned}
x_2(t) = S_2(t)\Bigg(& x_2(0) + A_{22}^{-1}(A_{21}x_1(0) + B_2u(0)) \\
&+ A_{22}^{-2}\left(A_{21}\frac{\mathrm{d}}{\mathrm{d}t}x_1(0) + B_2\frac{\mathrm{d}}{\mathrm{d}t}u(0)\right) \\
&- A_{22}^{-1}(A_{21}x_1(t) + B_2u(t)) - A_{22}^{-2}\left(A_{21}\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) + B_2\frac{\mathrm{d}}{\mathrm{d}t}u(t)\right)\Bigg) \\
&+ \int_0^t S_2(t-s)A_{22}^{-2}\left(A_{21}\frac{\mathrm{d}^2}{\mathrm{d}t^2}x_1(s) + B_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(s)\right)\mathrm{d}s. \qquad (3.8)
\end{aligned}$$

By truncating the transient and the last term in (3.8), and using $B_2 = 0$ we obtain from (3.1)

$$(I + A_{12}A_{22}^{-2}A_{21})\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = \tilde{A}_{11}x_1(t) + \tilde{B}_1u(t) \qquad (3.9)$$

and the corresponding approximate inertial manifold is given by

$$x_2 = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-2}A_{21}(I + A_{12}A_{22}^{-2}A_{21})^{-1}(\tilde{A}_{11}x_1 + \tilde{B}_1u).$$

Note that

$$\begin{aligned}
&|[(I_2 - \lambda A_{22}^{-1})^{-1} - (I_2 + \lambda A_{22}^{-1})]A_{22}^{-1}A_{21}x_1| \\
&\quad = \lambda^2\left|\sum_{i=0}^{\infty}\lambda^i A_{22}^{-i}\right||A_{22}^{-3}A_{21}x| \leqslant \bar{M}\lambda^2|A_{22}^{-3}A_{21}x_1|
\end{aligned}$$

for a constant $\bar{M}$ depending on λ and all $x_1 \in X_1$. Here, of course, it is assumed that $\|\lambda A_{22}^{-1}\| < 1$. It can be argued that the convergence rate of the eigenpair $(\lambda^N, (x^N, p^N))$ of (3.7) to the one of H is proportional to $\lambda^2\|A_{22}^{-3}A_{21}\|$ for each eigen-pair $(\lambda, (x, p))$ of H. Since $(x, \Pi x)$ is the eigen-function corresponding to the eigenvalue λ of H

$$p_i^N \text{ approximates } \Pi x_i^N, \quad x_i^N \in X_1$$

for each eigen-pair $(\lambda_i^N, (x_i^N, p_i^N))$ of (3.7). Based on this analysis we propose the following reduced-order algorithm for construction of the optimal feedback gain K.

Algorithm. Find the eigenpairs $(\lambda_i, (x_i, p_i))$ of (3.7), where the eigenvalues λ_i are ordered with respect to their real part. Form the matrices V and Y consisting of the first M vectors of x_i and p_i, respectively. Define $K^M\phi = B_1^*Y(V^*V)^{-1}(V, \phi)_X, \phi \in X$.

Here X_1 is identified with R^N and $\phi \to (V^*V)^{-1}(V, \phi)_X$ is the orthogonal projection of X onto X_1. M can be much smaller than N and thus the Algorithm offers a further order reduction for construction of the optimal feedback gain. It requires to find the ordered eigenpairs of (3.7) (not the nonnegative solution Π^N to the corresponding algebraic Riccati equation). If $M = N$, then YV^{-1} is the matrix representation of Π^N. Moreover, the span $\{x_i \in X_1, 1 \leqslant i \leqslant M\}$ approximates the closed invariant subspace of $A - BB^*\Pi$.

Remarks

(1) The procedure (3.3) to (3.8) can be repeated (by iteratively applying the integration by part formula) to obtain higher order approximations. As shown above this is equivalent to taking the higher order terms in the Neumann series of $(I - \lambda A_{22})^{-1}$.

(2) The reduced-order method we propose can be applied to higher-order control systems provided that we select X_1 so that $e^{A_{22}t}$ has a rapid exponential decay rate.

(3) The Algorithm requires to find sub-eigenspaces of the Hamiltonian matrix and thus appropriate variants of the QR method can be applied.

4. Numerical tests

In this section we demonstrate the applicability of the proposed reduced-order method using the one dimensional diffusion–advection equation

$$\frac{\partial}{\partial t} y(t,x) = \nu \frac{\partial^2}{\partial x^2} y + \frac{\partial}{\partial x} y + b(x)u(t), \quad x \in (0,1)$$

with homogeneous boundary conditions $y(t,0) = y(t,L) = 0$, where $\nu > 0$ and $b \in L^2(0,1)$ is the control distribution function. We let $X = L^2(0,1)$ and $U = R^1$ and define

$$A\phi = \nu \frac{d^2}{dx^2}\phi + \frac{d}{dx}\phi \quad \text{with } \operatorname{dom}(A) = H^2(0,1) \cap H_0^1(0,1),$$

and $Bu = b(x)u \in X$. Since

$$(A\phi, \phi) = -\nu \left| \frac{d}{dx}\phi \right|_X^2 \leqslant 0, \quad \text{for } \phi \in \operatorname{dom}(A)$$

it follows from the Lumer–Phillips theorem [17] that A generates a C_0 semigroup on X. Let

$$A_0\phi = \nu \frac{d^2}{dx^2}\phi \quad \text{with } \operatorname{dom}(A_0) = \operatorname{dom}(A).$$

Then A_0 is a negative, self-adjoint operator on X with eigenfunction $\phi_k = \sin(k\pi x)$, $k \geqslant 1$. We use the normalized eigen-family as the orthonormal basis and set $\nu = 0.2$, $b(x) = \sin(\pi x)$ and $Q = I$ for our numerical tests. Due to the boundary layer behavior of the solution near $x = 1$ this choice of the orthonormal basis may not work well for smaller $\nu > 0$. But for the choice $\nu = 0.2$ with $N = 200$ as below it works well. The following table summarizes our numerical findings.

Exact	Standard	0th Order
$-1.7878 + 1.1557i$	$-1.7878 + 1.1557i$	$-1.7848 + 1.1524i$
$-4.0034 + 0.6613i$	$-4.0033 + 0.6614i$	$-3.9235 + 0.7749i$
-6.4300	-6.4306	$-7.2564 + 0.5860i$
1st Order	**2nd Order**	**3rd Order**
$-1.7878 + 1.1557i$	$-1.7878 + 1.1556i$	$-1.7878 + 1.1557i$
$-4.0033 + 0.6613i$	$-4.0001 + 0.6648i$	$-4.0034 + 0.6612i$
-6.4316	-6.4609	-6.4301

The Table shows the first three closed-loop eigenvalues. 'Exact' refers to the eigenvalues obtained for the solution with the standard Galerkin procedure ($N = 200$), 'Standard' stands for the standard Galerkin approach with $N = 100$, and 0th, 1st, 2nd and, 3rd are for $N = 20$ with zeroth-, first-, second- and third-order reduced-order method, respectively. The second order reduced order method with $N = 20$ is comparable to the standard Galerkin approach with $N = 100$. Similar results are observed for the convergence of the feedback gains.

As stated in Remarks in Section 3, the proposed algorithms require to find sub-eigenspaces of the corresponding Hamiltonian matrix instead of computing the solution Π^N to the corresponding algebraic Riccati equation. For example we can use the "eigs"-routine in the Matlab to perform this task. So, it offers a significant reduction of the numerical effort compared to the standard Riccati equation based algorithm.

References

[1] J.A. Atwell, B.B. King, Proper orthogonal decomposition for reduced basis feedback controllers for parabolic equations, Math. Comput. Modell. 33 (2001) 1–19.

[2] R.F. Curtain, Model reduction for control design for distributed parameter systems, Directions in Distributed Parameter Systems, in: R. Smith, M. Demetriou (Eds.), Frontiers Appl. Math., vol. 27, SIAM, Philadelphia, 2003, pp. 95–121.

[3] R.F. Curtain, H. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer-Verlag, New York, 1995.

[4] C. Foias, O. Manley, R. Temam, Modeling of the interaction of small and large eddies in two dimensional turbulent flows, RAIRO Math. Mod. Numer. Anal. 22 (1988) 93–114.

[5] K. Glover, R.F. Curtain, J.R. Partington, Realization and approximation of linear infinite-dimensional systems with error bounds, SIAM J. Control Optim. 26 (1988) 863–898.

[6] K. Ito, Strong convergence and convergence rates of approximating solutions for algebraic Riccati equation in Hilbert spaces, in: Proceedings of Conference on Control and Identification of Distributed Parameter Systems, 6–12 July 1986, Vorau, Lecture Notes in Control and Info. Sciences, vol. 102, Springer-Verlag, 1987, pp. 153–166.

[7] K. Ito, K. Kunisch, Receding horizon optimal control for infinite dimensional systems, the Special Volume of COCV dedicated to J.L. Lions, ESAIM 8 (2002) 741–760.

[8] K. Ito, S.S. Ravindran, A reduced order method for simulation and control of fluid flows, J. Comput. Phys. 143 (1998) 403–425.

[9] K. Ito, S.S. Ravindran, Optimal control of thermally convected fluid flow, SIAM J. Sci. Comput. (1998) 1847–1869.
[10] K. Ito, J. Schroeter, Reduced order feedback synthesis for viscous incompressible flows, Math. Comput. Modell. 33 (2001) 173–192.
[11] E.A. Jonckheere, L.M. Silberman, A new set of invariants for linear systems—applications to reduced order compensator design, IEEE Trans. Automat. Controls 28 (1983) 953–964.
[12] K. Kunisch, S. Volkwein, Galerkin proper orthogonal decomposition methods for parabolic problems, Numer. Math. 90 (2001) 117–148.
[13] H.V. Ly, H.T. Tran, Proper orthogonal decomposition for flow calculation and optimal control in a horizontal CVD reactor, Quart. Appl. Math., in press.
[14] D.G. Meyer, Fractional balanced reduction: model reduction via fractional representation, IEEE Trans. Automat. Control 35 (1990) 1341–1345.
[15] M. Marion, R. Temam, Nonlinear Galerkin methods, SIAM J. Numer. Anal. 26 (1995) 894–914.
[16] R.J. Ober, D.C. MaFarlane, Balanced canonical forms for minimal systems: a normalized coprime factor approach, Linear Algebra Appl. 122 (1989) 23–64.
[17] A. Pazy, Semigroup of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.